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On the nonlinear reflection of a gravity wave at a critical level. Part 3

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In part 2 (Brown & Stewartson 1982) of this paper we set out the linearized solution of the critical layer that is expected to hold some time after the forced internal gravity wave of small amplitude ϵ is initiated at an infinite distance above the shear layer. This differed from that presented in part 1 (Brown & Stewartson 1980) in that in part 2 we exploited to a much greater extent the fact that the Richardson number J was large and obtained the solution in a form consisting of explicit functions rather than infinite integrals. Also it was demonstrated that at times $t = O(1)$ the wave did not penetrate beyond a certain level in the critical layer, and that critical-layer noise only was created above this line and transmitted below it. In this paper we examine the development of this solution on a longer time scale τ ($\propto \epsilon^{\frac{2}{3}}t$) and show how the reflection and transmission coefficients which are exponentially small,

$$O\{\exp(-(J-\tfrac{1}{4})^{\frac{1}{2}}\pi)\},$$

when $\tau = 0$ increase with time. As in part 1 we obtain a reflection coefficient for the first harmonic that is $O(\tau^3)$, and because of the simpler formulation of the linearized solution are able to obtain a reflected second harmonic. These harmonics appear as complementary functions that are induced by singularities in the particular integrals of the equations. It is shown that the interaction between the initial-noise term in the lower part of the critical layer and an induced-noise term at the sixth stage of the expansion will eventually lead to a transmitted wave. This appears at the ninth stage of the expansion and its transmission coefficient is $O(\tau^{12})$ though it is not explicitly calculated here.

1. Introduction

The study of part 2 (Brown & Stewartson 1982) completes the formulation of the solution of the linearized equations when a plane wave of small amplitude ϵ is maintained at $y = +\infty$ in a strongly stratified shear flow. In § 5 of that paper we obtained the solution in the neighbourhood of the critical layer at $y = 0$ that will result some time after the forcing is initiated. When $\alpha U'(0)yt > \nu$ and $\nu \gg 1$ the stream function consists of two terms as in (2; 5.16),† the first of which corresponds to the prescribed incident wave and the second of which we have called critical-layer (CL) noise. When $\alpha U'(0)yt < \nu$ only the CL-noise is present as may be seen from (2; 5.17). However, as t increases this linear theory eventually becomes inadequate as the horizontal component of velocity and the temperature become large. As in part 1 (Brown &

† We denote equation (5.16) of part 2 by (2; 5.16), etc.

Stewartson 1980) the time scale on which they do so is $O(\epsilon^{-\frac{2}{3}})$ and the critical layer then has thickness $O(\epsilon^{\frac{2}{3}})$. In this third paper we examine developments on this new time scale $\tau = O(1)$ $(\tau = \epsilon^{\frac{2}{3}}\alpha t)$, and show that although, as shown in part 1, when $t = O(1)$ the critical layer acts as an almost perfect absorber of the incident wave, on this longer time scale the critical layer starts to return some energy to the shear layer in the form of, first, reflected and, later, transmitted waves.

In §§2, 3 we set up the nonlinear equations of the critical layer to order ϵ and initiate expansions of the stream function and temperature in fractional powers of τ, retaining in each term where appropriate the leading 'wave' and 'noise' contribution for $\nu \gg 1$. In §4 we show how the interaction of these two contributions in the region $\eta > 1$ $(\eta = \epsilon^{-\frac{2}{3}}U'(0)y\tau/\nu)$, induces singularities at $\eta = \eta_0(=\frac{1}{4}(1+\sqrt{5})^2)$ that generate two additional complementary functions for $\eta > \eta_0$. One again represents CL-noise but the other leads to a reflected wave with reflection coefficient $O(\tau^3)$. This is a first harmonic and its coefficient was also calculated in part 1 for the special case of a hyperbolic velocity profile for the basic shear flow. In §5 a similar investigation gives a second harmonic with reflection coefficient $O(\tau^{\frac{9}{2}})$, and for the first time a complementary function is induced that extends to the region $\eta < 0$. This turns out to be of importance for the eventual appearance of a transmitted wave.

The general strategy of the expansion is explained in §6, where it is shown that eventually all harmonics will occur in the reflected wave. In §7 the development of the solution in the region $\eta < 1$ is discussed and it is demonstrated that interaction between the initial CL-noise of (2; 5.17) with an induced complementary function of order τ^7, that itself also represents noise, will in turn lead to a transmitted wave with transmission coefficient $O(\tau^{12})$, although its coefficient is not calculated explicitly. In §8 we present a summary and discussion of all three papers.

2. The nonlinear equations of the critical layer

In §5 of part 2 we developed the linearized theory in the neighbourhood of the critical layer at $y = 0$. The solution obtained there is the ultimate form that results a long time $(t \gg \nu \gg 1)$ after the forcing at $y = \infty$ is initiated. When $\alpha U'(0)yt > \nu$ the stream function consists of two terms as in (2; 5.16) the first of which corresponds to the prescribed incident wave and the second of which we have described as critical-level (CL) noise. When $\alpha U'(0)yt < \nu$ the CL-noise only is present as may be seen from (2; 5.17). However, as t increases this linear theory eventually becomes inadequate as the horizontal component of velocity and the temperature become large. As in part 1 the time scale on which they do so is $O(\epsilon^{-\frac{2}{3}})$ and the critical layer then has thickness $O(\epsilon^{\frac{2}{3}})$. If in (2; 2.1) we write

$$y = \epsilon^{\frac{2}{3}}Y/U'(0), \quad \alpha x = X, \quad \tau = \epsilon^{\frac{2}{3}}\alpha t,$$
$$\psi = \epsilon^{\frac{4}{3}}\Psi(X,Y,\tau)/U'(0), \quad T = \epsilon^{-\frac{1}{3}}R'(0)S(X,Y,\tau)/U'(0), \tag{2.1}$$

then the appropriate equations are, in the limit $\epsilon = 0$,

$$\left(\frac{\partial}{\partial\tau}+Y\frac{\partial}{\partial X}\right)\frac{\partial^2\Psi}{\partial Y^2}+(\nu^2+\tfrac{1}{4})\frac{\partial S}{\partial X} = \frac{\partial\Psi}{\partial X}\frac{\partial^3\Psi}{\partial Y^3}-\frac{\partial\Psi}{\partial Y}\frac{\partial^3\Psi}{\partial X\,\partial Y^2}, \tag{2.2a}$$

$$\left(\frac{\partial}{\partial\tau}+Y\frac{\partial}{\partial X}\right)S-\frac{\partial\Psi}{\partial X} = \frac{\partial\Psi}{\partial X}\frac{\partial S}{\partial Y}-\frac{\partial\Psi}{\partial Y}\frac{\partial S}{\partial X}. \tag{2.2b}$$

The initial conditions for these equations are specified by the requirement that the solution shall match, as $\tau \to 0$, with the linearized solution of § 5 of part 2, which, as noted above, we envisage to be the form, when $\nu \gg 1$, taken for large t of the solution of an initial-value problem starting at $t = 0$. Then $\tau = O(1)$ represents the next stage in the development of the flow and we shall find that on this time scale the outer flow, where $y = O(1)$, although still linear, is no longer steady.

For our initial conditions for (2.2) we shall take the solution summarized in (2; 5.16), (2; 5.17), which means that

$$\Psi = e^{iX}\Psi_{11}(Y,\tau)+\text{c.c.}, \quad S = \frac{i}{\nu}e^{iX}\frac{\partial \Psi_{11}}{\partial Y}+\text{c.c.} \tag{2.3}$$

as $\tau \to 0$, where

$$\Psi_{11} \sim A\tau^{-\frac{1}{2}+i\nu}\left\{\eta^{\frac{1}{2}-i\nu}+\frac{\beta e^{-i\nu\eta}}{\nu^{\frac{1}{2}}(\eta-1)}\right\}, \tag{2.4}$$

when $\eta > 1$, and

$$\Psi_{11} \sim A\frac{\beta\tau^{-\frac{1}{2}+i\nu}e^{-i\nu\eta}}{\nu^{\frac{1}{2}}(\eta-1)}, \tag{2.5}$$

when $\eta < 1$. The meaning of $\sim$ in (2.4), (2.5) and subsequently in this paper is that the terms given explicitly on the right-hand side are leading terms in an asymptotic expansion of the left-hand side in descending powers of ν. When two terms appear there are two components to the expansion of different forms each of which is dominated by the term explicitly quoted. Here

$$\eta = \frac{\alpha U'(0)\,yt}{\nu} = \frac{Y\tau}{\nu}, \quad Ae^{\frac{3}{2}i\nu} = a_-, \quad \beta = (2\pi)^{-\frac{1}{2}}e^{i\nu+\frac{3}{4}i\pi}, \tag{2.6}$$

where a_- is defined in (2; 5.14). Near $\eta = 1$

$$\Psi_{11} \sim -i2^{\frac{1}{2}}A\beta\tau^{-\frac{1}{2}+i\nu}e^{-i\nu\eta+i\zeta^2}\int_{-\infty}^{\zeta} e^{-i\zeta_1^2}d\zeta_1, \tag{2.7}$$

where, as in (2; 5.7),

$$\zeta = (\eta-1)\left(\tfrac{1}{2}\nu\right)^{\frac{1}{2}}. \tag{2.8}$$

In addition there will be matching conditions with the region outside the critical layer, namely that the incident wave from above cannot be changed and there can be no incident wave from below. The reflected and transmitted waves have amplitudes that are functions of τ and ultimately include all harmonics. These solutions satisfy the linearized equations with $\partial/\partial t$ set equal to zero, and have been derived in § 4 of part 2.

We now proceed as in part 1, setting up an expansion for Ψ, S by repeated substitution into (2.2), beginning with Ψ_{11}, S_{11}. We write

$$\Psi = \sum_{r=1}^{\infty}\Psi_r(X,Y,\tau), \quad S = \sum_{r=1}^{\infty}S_r(X,Y,\tau), \tag{2.9}$$

where Ψ_r, S_r are of the form

$$\Psi_r = \sum_{n=-r}^{r}e^{niX}\Psi_{rn}(Y,\tau), \quad S_r = \sum_{n=-r}^{r}e^{niX}S_{rn}(Y,\tau), \tag{2.10}$$

$$\Psi_{10} = S_{10} = 0, \quad S_{11} = \frac{i}{\nu}\frac{\partial \Psi_{11}}{\partial Y}. \tag{2.11}$$

Also

$$\tilde{\Psi}_{rn} = \Psi_{r,-n}, \quad \tilde{S}_{rn} = S_{r,-n}, \tag{2.12}$$

the complex conjugates being denoted by tildes, and where without loss of generality we may take $r-n$ to be an even integer or zero.

On substituting (2.9), (2.10) into (2.2) we find that Ψ_{rn}, S_{rn} satisfy

$$\left(\frac{\partial}{\partial\tau} + inY\right)\frac{\partial^2\Psi_{rn}}{\partial Y^2} + in(\nu^2+\tfrac{1}{4})\,S_{rn} = \frac{\partial M_{rn}}{\partial Y}, \tag{2.13}$$

$$\left(\frac{\partial}{\partial\tau} + inY\right)S_{rn} - in\Psi_{rn} = H_{rn}, \tag{2.14}$$

where M_{rn}, H_{rn} are known functions of Y and τ depending on the previously calculated Ψ_p, S_p with $1 \leqslant p \leqslant r-1$. If we eliminate S_{rn} between (2.13), (2.14) we find that the resulting operator on Ψ_{rn} can be factorized as could that on Φ in (2; 5.1), and we may write

$$\Psi_{rn} = \frac{1}{2\nu}(G^-_{rn} - G^+_{rn}), \quad S_{rn} = \frac{1}{2\nu}\frac{\partial}{\partial Y}\left(\frac{G^-_{rn}}{\frac{1}{2}-i\nu} - \frac{G^+_{rn}}{\frac{1}{2}+i\nu}\right), \tag{2.15}$$

where

$$\left(\frac{\partial}{\partial\tau} + niY\right)\frac{\partial G^+_{rn}}{\partial Y} - ni(\tfrac{1}{2}+i\nu)\,G^+_{rn} = i\{(\tfrac{1}{2}+i\nu)\,M_{rn} - (\tfrac{1}{4}+\nu^2)\,H_{rn}\}, \tag{2.16}$$

$$\left(\frac{\partial}{\partial\tau} + niY\right)\frac{\partial G^-_{rn}}{\partial Y} - ni(\tfrac{1}{2}-i\nu)\,G^-_{rn} = i\{(\tfrac{1}{2}-i\nu)\,M_{rn} - (\tfrac{1}{4}+\nu^2)\,H_{rn}\}. \tag{2.17}$$

As we found in part 1 each Ψ_{rn}, $\tau^{-1}S_{rn}$ is of the form

$$\tau^{-2+ni\nu+\frac{3}{2}r}A^{\frac{1}{2}(r+n)}\tilde{A}^{\frac{1}{2}(r-n)}\mathscr{F}_{rn}(\eta,\nu), \tag{2.18}$$

where the powers of τ can differ from those associated with the reflection and transmission coefficients $\mathscr{R}_{rn}$ and $\mathscr{T}_{rn}$ because of the τ-dependence of η. Part of our aim in the rest of this paper is to obtain, at least in principle, the leading term in the uniform expansion of $\mathscr{F}$ in descending powers of ν, a typical term being of the form

$$\nu^{-\frac{1}{2}N}g_1(\eta,N)\,e^{i\nu f_1(\eta)}, \tag{2.19}$$

where N is an integer, f_R $(R = 1, 2, \ldots)$ is a function of η only, and $g_R(\eta, N)$ a function of η and N. Broadly, we shall find that

$$\mathscr{F}_{rn} = O(\nu^{-1}\mathscr{F}_{r-1,m}) \tag{2.20}$$

for all admissible n, m but exceptions will occur because of cancellations and the (highly significant) generation of complementary functions.

In fact, were it not for these, the expansion (2.9) would be straightforward and of little interest. They arise in the following way. From (2.18) we may expect a typical term of M_{rn}, H_{rn} to be

$$\nu^{-\frac{1}{2}N}g_2\,e^{i\nu f_2}, \tag{2.21}$$

and to force a G_{rn} of the form

$$\nu^{-\frac{1}{2}N}\frac{g_2 g_3}{K(f_2')}e^{i\nu f_2}, \tag{2.22}$$

where K is a quadratic in f_2' and essentially also in η. If and when K vanishes we say *resonance* has occurred, and to prevent the singularity which would otherwise arise at this point a complementary function must be added to G_{rn}. The reflected and transmitted waves can only be set up by these functions. We note that, as in part 2, the terms that will generate reflected and transmitted waves in the outer flow have been separated in (2.16), (2.17). For $|Y| \gg 1$ it is G_{rn}^{-} that will have a complementary function proportional to $|Y|^{\frac{1}{2}-i\nu}$, which corresponds to incident waves from above the layer, of which there is to be only one, with $n = 1$, and to transmitted waves below the layer. On the other hand, G_{rn}^{+} has a complementary function proportional to $|Y|^{\frac{1}{2}+i\nu}$ and so corresponds to waves reflected above the layer, and to waves incident from below the layer, which must be excluded.

3. Calculation of the second-order terms

It emerges that, as far as the calculation of $\mathscr{R}_{31}, \mathscr{R}_{42}$, the first two non-zero reflection coefficients as defined in (2; 4.14), is concerned, we do not need the explicit form of any terms for $\eta < 1$ or for $\eta \simeq 1$. However, we shall find Ψ_{20} for all η in order to illustrate the effect of the incident wave on the mean flow. First we calculate Ψ_{20}, S_{20} in

$$\Psi_2 = e^{2iX}\Psi_{22} + \Psi_{20} + e^{-2iX}\tilde{\Psi}_{22}, \quad S_2 = e^{2iX}S_{22} + S_{20} + e^{-2iX}\tilde{S}_{22}. \tag{3.1}$$

It follows from (2.2) that, if $\partial\Psi_{20}/\partial\tau, \partial S_{20}/\partial\tau$ are bounded as $|Y| \to \infty$,

$$\frac{\partial\Psi_{20}}{\partial\tau} = i\left(\Psi_{11}\frac{\partial\tilde{\Psi}_{11}}{\partial Y} - \tilde{\Psi}_{11}\frac{\partial\Psi_{11}}{\partial Y}\right), \quad \frac{\partial S_{20}}{\partial\tau} = i\frac{\partial}{\partial Y}\left(\Psi_{11}\tilde{S}_{11} - \tilde{\Psi}_{11}S_{11}\right), \tag{3.2}$$

where Ψ_{11}, S_{11} are given by (2.3)–(2.8), and the additive arbitrary functions of τ have been set equal to zero as they do not affect the subsequent discussion. The right-hand sides of these equations take different forms in the three regions, and when $\eta < 1$ we obtain, on integration,

$$\Psi_{20} \sim \frac{2|A\beta|^2\tau}{\nu(\eta-1)}, \quad S_{20} \sim \frac{|A\beta|^2\tau^2(\eta-3)}{\nu^4(\eta-1)^3}, \tag{3.3}$$

to which should be added complementary functions $a_{20}\tau/\eta$, $b_{20}\tau^2/\eta^2$, where a_{20}, b_{20} are constants since both complementary functions must depend on Y alone, and in the τ, η variables must have τ-dependence τ and τ^2, respectively, to satisfy the requirements of (2.18) on Ψ_{20}, S_{20}. However, $a_{20} = b_{20} = 0$ to avoid singularities at $Y = 0$. In the transition zone near $\eta = 1, \zeta = O(1)$ and the solution that matches with (3.3) is

$$\Psi_{20} \sim -2\left(\frac{2}{\nu}\right)^{\frac{1}{2}}\frac{|A|^2\tau}{\pi}\{\zeta|I(\zeta)|^2 - \tfrac{1}{2}ie^{-i\zeta^2}\tilde{I}(\zeta) + \tfrac{1}{2}ie^{i\zeta^2}I(\zeta)\}, \tag{3.4a}$$

$$S_{20} \sim \frac{|A|^2\tau^2}{2^{\frac{1}{2}}\nu^{\frac{5}{2}}\pi}\frac{d}{d\zeta}\{|I(\zeta)|^2\}, \tag{3.4b}$$

where

$$I(\zeta) = \int_{-\infty}^{\zeta} e^{-i\zeta_1^2}d\zeta_1. \tag{3.5}$$

When $\eta > 1$ the result corresponding to (3.3) is

$$\Psi_{20} \sim -|A|^2\tau\left\{2 - \frac{2}{\eta} + \frac{i(\eta+1)}{\nu^{\frac{3}{2}}\eta^{\frac{1}{2}}(\eta-1)^2}(\beta\eta^{i\nu}e^{-i\nu\eta} - \tilde{\beta}\eta^{-i\nu}e^{i\nu\eta})\right\}, \tag{3.6a}$$

$$S_{20} \sim -\frac{|A|^2\tau^2 i}{\nu^{\frac{5}{2}}\eta^{\frac{3}{2}}}(\beta\eta^{i\nu}e^{-i\nu\eta} - \tilde{\beta}\eta^{-i\nu}e^{i\nu\eta}). \tag{3.6b}$$

In making the choice $a_{20} = 2|A|^2$, $b_{20} = 0$ for $\eta > 1$ two arguments may be used. Either they are necessary to ensure a detailed match with (3.4) as $\zeta \to \infty$, or we may note that, in terms of ζ, $\Psi_{20} = O(\nu^{-\frac{1}{2}})$, $S_{20} = O(\nu^{-\frac{3}{2}})$ from (3.3), and so (3.6) must lead to similar orders of magnitude as $\eta \to 1+$. The second argument is simpler to apply and may be used in more complicated situations to explain the appearance of complementary functions and hence travelling waves in parts of the critical layer. The term $O(\tau/\eta)$ in Ψ_{20} for large $|\eta|$ was also obtained in part 1, and implies that the horizontal velocity component contains an adjustment to the mean profile that is $O(\epsilon^2 y^{-2})$.

To evaluate Ψ_{22}, S_{22} we return to (2.16), (2.17) and find that, to leading order in ν,

$$M_{22} \sim -i\nu H_{22} = i\left\{\Psi_{11}\frac{\partial^2 \Psi_{11}}{\partial Y^2} - \left(\frac{\partial \Psi_{11}}{\partial Y}\right)^2\right\}, \tag{3.7}$$

so that the equation for G_{22}^+ is homogeneous, and hence $G_{22}^+ = 0$ as there must be no incident wave from below the critical layer. Although here we have obtained this result only to the leading-order term in ν, in part 1 it was shown to be true for all ν. If in (2.17) we now write $G_{rn}^- = 2\nu\Psi_{rn}$ it becomes

$$\left(\frac{\partial}{\partial \tau} + 2iY\right)\frac{\partial \Psi_{22}}{\partial Y} - 2i(\tfrac{1}{2} - i\nu)\Psi_{22} = i\left\{\Psi_{11}\frac{\partial^2 \Psi_{11}}{\partial Y^2} - \left(\frac{\partial \Psi_{11}}{\partial Y}\right)^2\right\}, \tag{3.8}$$

where the right-hand side has been retained to leading order in ν. We shall calculate Ψ_{22} in $\eta > 1$ only, as it will be shown in § 4 that its precise value when $\eta < 1$ or $\eta \simeq 1$ is not required to calculate the reflection or transmission coefficients, at least to the order considered here. Using (2.4) and keeping a contribution of the type of the first term of (2.4) (the 'wave' term) and one of the type of the second term (the CL-noise term), (3.8) becomes, to leading order in ν,

$$\left(\frac{\partial}{\partial \tau} + 2iY\right)\frac{\partial \Psi_{22}}{\partial Y} - 2i(\tfrac{1}{2} - i\nu)\Psi_{22} \sim A^2 i\tau^{1+2i\nu}\left\{\frac{i}{\nu}\eta^{-1-2i\nu} - \frac{\beta(\eta-1)\,e^{-i\nu\eta}}{\nu^{\frac{1}{2}}\eta^{\frac{3}{2}+i\nu}}\right\} \tag{3.9}$$

for $\eta > 1$, where the first term on the right-hand side is $O(\nu^{-1})$ instead of the expected $O(1)$ owing to a cancellation. Again to leading order in ν, the solution of (3.9) is

$$\Psi_{22} \sim A^2 i\tau^{1+2i\nu}\left\{\frac{i}{2\nu^2}\eta^{-1-2i\nu} - \frac{\beta(\eta-1)\,\eta^{-\frac{1}{2}-i\nu}e^{-i\nu\eta}}{\nu^{\frac{3}{2}}(\eta^2+1)}\right\}. \tag{3.10}$$

The corresponding solution for S_{22} in $\eta > 1$ is

$$S_{22} \sim \frac{i}{\nu}\frac{\partial \Psi_{22}}{\partial Y} \sim A^2\frac{i}{\nu}\tau^{2+2i\nu}\left\{\frac{i}{\nu^2}\eta^{-2-2i\nu} - \frac{\beta(\eta^2-1)\,\eta^{-\frac{3}{2}-i\nu}e^{-i\nu\eta}}{\nu^{\frac{3}{2}}(\eta^2+1)}\right\}. \tag{3.11}$$

In $\eta < 1$ both Ψ_{22} and $\tau^{-1}S_{22}$ are made up of noise terms only that consist of a product of $\tau^{1+2i\nu}e^{-2i\nu\eta}$ with an algebraic function of η, and this is the only property we shall need. When $\eta \simeq 1$ there is a boundary-layer solution that bridges these, but again it is passive and we shall not require its precise form.

The absence of a term in $|\eta|^{\frac{1}{2}-i\nu}$ or $|\eta|^{\frac{1}{2}+i\nu}$ indicates that there is no transmitted or reflected second harmonic at this stage. The first reflected second harmonic occurs for $\eta > 1$ at the Ψ_{42} stage. In § 4 we calculate the first correction $\mathscr{R}_{31}$ to the reflection coefficient. In part 1 the transmission coefficient $\mathscr{T}_{31}$ was shown to be exponentially small in ν.

4. The calculation of $\mathscr{R}_{31}(\tau)$

For this purpose we must solve (2.16), (2.17) for $G_{31}^{\pm}$, the forcing terms being

$$-\nu(M_{31}+i\nu H_{31}) = i\nu\left(-2\frac{\partial\tilde{\Psi}_{11}}{\partial Y}\frac{\partial\Psi_{22}}{\partial Y}-4\frac{\partial^2\tilde{\Psi}_{11}}{\partial Y^2}\Psi_{22}-i\nu\Psi_{11}\frac{\partial S_{20}}{\partial Y}-\Psi_{11}\frac{\partial^2\Psi_{20}}{\partial Y^2}\right), \quad (4.1)$$

$$\nu(M_{31}-i\nu H_{31}) = i\nu\left(-2\tilde{\Psi}_{11}\frac{\partial^2\Psi_{22}}{\partial Y^2}-4\frac{\partial\tilde{\Psi}_{11}}{\partial Y}\frac{\partial\Psi_{22}}{\partial Y}+\Psi_{11}\frac{\partial^2\Psi_{20}}{\partial Y^2}-i\nu\Psi_{11}\frac{\partial S_{20}}{\partial Y}-2\frac{\partial\Psi_{11}}{\partial Y}\frac{\partial\Psi_{20}}{\partial Y}\right), \quad (4.2)$$

respectively. From an examination of the leading terms in the asymptotic expansions of the right-hand sides of (4.1), (4.2) we see that Ψ_{31} is forced by a number of terms of the form

$$\tau^{-2+ni\nu+\frac{3}{2}r}e^{-i\nu\lambda\eta}\eta^{-i\nu(n-\lambda)}\Phi_{rn\lambda}^{\pm}(\eta), \quad (4.3)$$

where λ is any integer satisfying $-\frac{1}{2}(r-n)\leqslant\lambda\leqslant\frac{1}{2}(r+n)$ when $\eta>1$, and $\lambda = n$ when $\eta<1$; also $r=3, n=1$ and $\Phi_{rn\lambda}$ is a known algebraic function of η† which vanishes as $|\eta|\to\infty$. A corresponding result holds for general r, n but there may be additional terms which arise from complementary functions of (2.16), (2.17). We have already seen how these arise when $r=2, n=0$, and another set arises at this stage. We now write

$$G_{rn}^{\pm} = \tau^{-2+ni\nu+\frac{3}{2}r}e^{-i\nu\lambda\eta}\eta^{-i\nu(n-\lambda)}T_{rn}^{\pm}(\eta), \quad (4.4)$$

in (2.16), (2.17), so that $T_{rn}^{\pm}$ satisfy

$$\frac{\eta}{\nu}\frac{d^2T_{rn}^{\pm}}{d\eta^2}+\left\{i(n-2\lambda)(\eta-1)+\frac{3r-2}{2\nu}\right\}\frac{dT_{rn}^{\pm}}{d\eta}+\left\{\frac{\nu}{\eta}Q_{rn\lambda}^{\pm}(\eta)-\tfrac{1}{2}in-i\lambda(\tfrac{3}{2}r-1)-\frac{i}{\eta}(n-\lambda)(\tfrac{3}{2}r-2)\right\}T_{rn}^{\pm} = \Phi_{rn\lambda}^{\pm}, \quad (4.5)$$

where $Q_{rn\lambda}^{\pm}(\eta)$ is the quadratic

$$Q_{rn\lambda}^{\pm}(\eta) = \lambda(n-\lambda)\eta^2+\{n^2-2\lambda(n-\lambda)\pm n\}\eta+\lambda(n-\lambda). \quad (4.6)$$

Our requirements on G_{31} are that G_{31}^{+} must have no term $O(|\eta|^{\frac{1}{2}+i\nu})$ when η is large and negative, for if so there would be a wave incident on the shear layer from below, and that G_{31}^{-} must not include a term $O(\eta^{\frac{1}{2}-i\nu})$ when η is large and positive or there would be an additional wave $O(\tau^3)$ incident from above. Also, $G_{rn}^{\pm}$ must not be singular at $\eta=0$. This implies, from (4.4), that, for small η, $T_{rn}\sim\eta^{i\nu(n-\lambda)}$. This is a property of one of the solutions of (4.5); the other must therefore be excluded in any region including $\eta=0$.

The complementary functions of (4.5) for T_{rn}^{+} are of the asymptotic form

$$(e^{i\nu\lambda\eta}\eta^{i\nu(n-\lambda)})\,\eta^{\frac{1}{2}+i\nu},\quad (e^{i\nu\lambda\eta}\eta^{i\nu(n-\lambda)})\,e^{-i\nu n\eta}\eta^{-i\nu(n+1)-\frac{1}{2}(3r-1)}, \quad (4.7)$$

as $|\eta|\to\infty$, while for T_{rn}^{-} they are of the form

$$(e^{i\nu\lambda\eta}\eta^{i\nu(n-\lambda)})\,\eta^{\frac{1}{2}-i\nu},\quad (e^{i\nu\lambda\eta}\eta^{i\nu(n-\lambda)})\,e^{-i\nu n\eta}\eta^{-i\nu(n-1)-\frac{1}{2}(3r-1)}, \quad (4.8)$$

as $|\eta|\to\infty$. When $\eta\gg1$ the first complementary function in (4.7) corresponds to a

† I.e. in which the powers of η involved are independent of ν.

reflected wave in G^+_{rn} and the second decays and corresponds to noise. In (4.8) the first corresponds to a wave incident from above the layer, which must be excluded, and the second is again noise. When $\eta \ll -1$ the first complementary function in (4.7) is unacceptable, while the first in (4.8) gives a transmitted wave in G^-_{rn}. Again the other terms decay. We shall see below how such complementary functions can be induced in the case $r = 3, n = 1$.

If the quadratic $Q^{\pm}_{rn\lambda}(\eta)$ in (4.6) has no zero for any η then for $\nu \gg 1$ the solution of (4.5) is

$$T^{\pm}_{rn} \sim \eta\Phi^{\pm}_{rn\lambda}(\eta)/\nu Q^{\pm}_{rn\lambda}(\eta), \tag{4.9}$$

so that $T^{\pm}_{rn} \to 0$ as $|\eta| \to \infty$. Let us now look at the implications of this result for T^-_{31}. If $\lambda = -1$ or 2, $Q^-_{31\lambda}(\eta) = -2(\eta-1)^2$ and has a double zero at $\eta = 1$, so that a particular integral for T^-_{31} is

$$\frac{-\eta\Phi^-_{31\lambda}(\eta)}{2\nu(\eta-1)^2}, \tag{4.10}$$

for any $\eta \neq 1$. However, on examining the behaviour of the solution near the possible singularity at $\eta = 1$ it may easily be shown that it is smoothed out in a region of thickness $O(\nu^{-\frac{1}{2}})$ and generates complementary functions for T^-_{31} of the form of the second of (4.8) for both $\eta < 1$ and $\eta > 1$. This represents noise and we are already in possession of such a term in G^-_{31}, since its exponential behaviour as a contribution to G^-_{31} is $e^{-i\nu\eta}$, which is present in G^-_{31} as the term of the form (4.3) with $\lambda = 1$. Again, if $\lambda = 0$ or 1, $Q^-_{31\lambda}(\eta) \equiv 0$, and (4.5) reduces to a first-order equation whose solution also has a singularity at $\eta = 1$ to be smoothed out in the region of thickness $O(\nu^{-\frac{1}{2}})$; but again no new exponential behaviour is added to G^-_{31} for either $\eta > 1$ or $\eta < 1$. Thus the exponential behaviours of the contributions to G^-_{31} are exactly those of its forcing terms, namely $e^{-i\nu\lambda\eta}\eta^{-i\nu(1-\lambda)}$ for $-1 \leqslant \lambda \leqslant 2$ if $\eta > 1$, and $\lambda = 1$ if $\eta < 1$.

The solution for T^+_{31} is quite different. If $\lambda = 0$ or 1, $Q^+_{31\lambda}$ is proportional to η and this does not lead to a singularity of (4.9). However, if $\lambda = -1$ or 2,

$$Q^+_{31\lambda}(\eta) = -2(\eta^2 - 3\eta + 1), \tag{4.11}$$

which vanishes when $\eta = \eta_0\,(>1)$ and $\eta = \eta_0^{-1}\,(<1)$, where

$$\eta_0 = \omega_0^2, \quad \omega_0 = \tfrac{1}{2}(1+\sqrt{5}). \tag{4.12}$$

The root that is less than unity is of no interest here, since when $\eta < 1$ the function $\Phi_{31\lambda}(\eta)$ is non-zero only if $\lambda = 1$. We first consider $\lambda = -1$ and show that there is a complementary function induced for $\eta > \eta_0$ and that it does indeed correspond to a reflected wave.

When $\lambda = -1$, equation (4.5) for T^+_{31} becomes

$$\frac{\eta}{\nu}\frac{d^2T^+_{31}}{d\eta^2} + \left\{3i(\eta-1) + \frac{7}{2\nu}\right\}\frac{dT^+_{31}}{d\eta} + \left\{-\frac{2\nu}{\eta}(\eta^2 - 3\eta + 1) + 3i - \frac{5i}{\eta}\right\}T^+_{31} = g_{-1}(\eta), \tag{4.13}$$

where for convenience we have written $\Phi^+_{31\lambda}(\eta) = g_\lambda(\eta)$. In the neighbourhood of $\eta = \eta_0$ we write

$$\eta - \eta_0 = \nu^{-\frac{1}{2}}p, \tag{4.14}$$

so that (4.13) reduces, when $\nu \gg 1$, to

$$3i\omega_0\frac{dT^+_{31}}{dp} - 2\frac{5^{\frac{1}{2}}}{\omega_0^2}pT^+_{31} = \nu^{-\frac{1}{2}}g_{-1}(\eta_0), \tag{4.15}$$

with solution

$$T_{31}^{+} \sim \frac{-ig_{-1}(\eta_0)}{3\nu^{\frac{1}{2}}\omega_0} e^{-i\gamma p^2} \int_{-\infty}^{p} e^{i\gamma p_1^2} dp_1 + \beta_{31} e^{-i\gamma p^2}, \tag{4.16}$$

where β_{31} is an arbitrary constant and $\gamma = 5^{\frac{1}{2}}/3\omega_0^3$. Now outside the region $p = O(1)$ a particular integral for T_{31}^{+} is

$$T_{31}^{+} \sim \frac{-\eta g_{-1}(\eta)}{2\nu(\eta^2 - 3\eta + 1)}, \tag{4.17}$$

and it is clear from (4.16) that these cannot match as $\eta \to \eta_0$ and as p tends to both plus and minus infinity, for any value of β_{31}. The remedy is to correct (4.17) by the addition of a complementary function $O(\nu^{-\frac{1}{2}})$. We write it in the form

$$C_{31}(\eta)\, e^{i\nu D_{31}(\eta)} [1 + O(\nu^{-\frac{1}{2}})], \tag{4.18}$$

and on substituting into (4.13) find that

$$\eta^2 D_{31}'^2 + 3\eta(\eta - 1) D_{31}' + 2(\eta^2 - 3\eta + 1) = 0, \tag{4.19}$$

$$\{2\eta D_{31}' + 3(\eta - 1)\} C_{31}' + \left(\eta D_{31}'' + \tfrac{7}{2} D_{31}' + 3 - \frac{5}{\eta}\right) C_{31} = 0. \tag{4.20}$$

Since (4.18) must provide a match with the term $e^{-i\gamma p^2}$ in (4.16), $D_{31}'(\eta) = 0$ when $\eta = \eta_0$, and hence

$$2\eta D_{31}' = -3(\eta - 1) + (\eta^2 + 6\eta + 1)^{\frac{1}{2}}, \tag{4.21}$$

is the required root of (4.19). It then follows from (4.20) that

$$C_{31}(\eta) = C_0 \frac{\{\eta + 3 + (\eta^2 + 6\eta + 1)^{\frac{1}{2}}\}^{\frac{9}{4}}}{(\eta^2 + 6\eta + 1)^{\frac{1}{4}} \{1 + 3\eta + (\eta^2 + 6\eta + 1)^{\frac{1}{2}}\}^{\frac{5}{4}}}, \tag{4.22}$$

where C_0 is a constant for $\eta > \eta_0$ and $\eta < \eta_0$, but not necessarily having the same value in each region. On integrating (4.21) we obtain

$$D_{31}(\eta) = -\tfrac{3}{2}\eta + 2\log\eta + \tfrac{1}{2}(\eta^2 + 6\eta + 1)^{\frac{1}{2}} + \tfrac{3}{2}\log\{\eta + 3 + (\eta^2 + 6\eta + 1)^{\frac{1}{2}}\} - \tfrac{1}{2}\log\{1 + 3\eta + (\eta^2 + 6\eta + 1)^{\frac{1}{2}}\}. \tag{4.23}$$

For $\eta > 0$ the obvious meaning may be attached to all terms of (4.23) and so, when $\eta \gg 1$,

$$D_{31}(\eta) = -\eta + 3\log\eta + \tfrac{3}{2} + \tfrac{1}{2}\log 2 + o(1). \tag{4.24}$$

Hence

$$G_{31}^{+}(\eta) \sim 2^{\frac{1}{2}i\nu - \frac{1}{4}} e^{\frac{3}{2}i\nu} C_0 \tau^{\frac{5}{2} + i\nu} \eta^{\frac{1}{2} + i\nu} \quad \text{as} \quad \eta \to \infty, \tag{4.25}$$

and corresponds to a reflected wave.

At $\eta = 0$, D_{31} has a logarithmic singularity, but the corresponding contribution to G_{31}^{+} remains smooth. A standard transition point (of the type associated with Airy functions) occurs when $\eta = -3 + \sqrt{8}$ $(\eta^2 + 6\eta + 1 = 0)$, out of which two branches emerge for $\eta < -3 + \sqrt{8}$; initially these are equal but one of them increases exponentially as η decreases further, while the other decreases exponentially. Only the first is therefore significant at the second transition point where $\eta = -3 - \sqrt{8}$, and in turn it splits into two branches defined by (4.23) in one of which $(\eta^2 + 6\eta + 1)^{\frac{1}{2}}$ is positive and in the other $(\eta^2 + 6\eta + 1)^{\frac{1}{2}}$ is negative for $\eta < -3 - \sqrt{8}$. The branch with $(\eta^2 + 6\eta + 1)^{\frac{1}{2}}$ negative corresponds to a wave incident from below that is exponentially large $(e^{\nu\pi})$ in magnitude when compared with Ψ_{11} and must be rejected. It follows that $C_0 = 0$

if $\eta < \eta_0$, and (4.18) matches with (4.16) if $\beta_{31} = 0$ and the value of C_0 in $\eta > \eta_0$ is such that

$$C_{31}(\eta_0)\, e^{i\nu D_{31}(\eta_0)} = 5^{-\frac{1}{4}} e^{-\frac{1}{4}i\pi} \left(\frac{\pi\omega_0}{3\nu}\right)^{\frac{1}{2}} g_{-1}(\eta_0). \tag{4.26}$$

The function $g_{-1}(\eta)$ is the coefficient of $\tau^{\frac{5}{2}+i\nu}\eta^{-2i\nu}e^{i\nu\eta}$ in (4.1) and the contribution to it from the terms involving Ψ_{22} is $O(\nu^{-\frac{3}{2}})$ and from those involving S_{20}, Ψ_{20} is $O(\nu^{-\frac{1}{2}})$. Hence ignoring the former we obtain, for $\eta > 1$,

$$g_{-1}(\eta) \sim -\frac{2A|A|^2\bar{\beta}}{\nu^{\frac{1}{2}}\eta}, \tag{4.27}$$

and from (4.22), (4.33), (4.26)

$$C_0 = 5^{-\frac{1}{4}} 2^{-\frac{11}{4}-\frac{5}{2}i\nu} \omega_0^{-5i\nu-\frac{1}{2}} e^{\frac{1}{2}i\nu} \nu^{-1} A|A|^2. \tag{4.28}$$

It follows from (4.25) that

$$\Psi_{31} \sim -\frac{e^{2i\nu}\, 2^{3i\nu-\frac{7}{2}} A|A|^2}{5^{\frac{1}{4}}(1+\sqrt{5})^{5i\nu+\frac{1}{2}}\nu^2} \tau^{\frac{5}{2}+i\nu}\eta^{\frac{1}{2}+i\nu}, \tag{4.29}$$

for $\eta \gg 1$. Then on use of (2; 4.12), (2; 5.14) and (2.1) we obtain for $\mathscr{R}_{31}(\tau)$, the first reflection coefficient, the expression

$$\mathscr{R}_{31}(\tau) \sim -\frac{2^{3i\nu-\frac{7}{2}} e^{-\frac{4}{3}i\nu} e^{2i\nu(1-\Gamma^+)}}{5^{\frac{1}{4}}(1+\sqrt{5})^{5i\nu+\frac{1}{2}}\nu^{1+2i\nu}} \left(\frac{R'(\infty)}{R'(0)}\right)^{\frac{1}{2}} \frac{(U'(0))^{2+2i\nu}}{U(\infty)} \tau^{3+2i\nu}, \tag{4.30}$$

which reduces to equation (9.32) of part 1 when $R'(\infty) = R'(0) = U'(0) = U(\infty) = 1$ and $\Gamma^+ = \log 2$.

The effect of the forcing term $\eta^{i\nu}e^{-2i\nu\eta}$ corresponding to $\lambda = 2$ can be discussed in the same way. A complementary function has to be added in $\eta > \eta_0$ and its exponential behaviour is $e^{-i\nu D_{31}(\eta)}$, where D_{31} is given in (4.23). For $\eta \gg 1$ the product

$$\eta^{i\nu} e^{-2i\nu\eta} e^{-i\nu D_{31}(\eta)} \approx \eta^{-2i\nu} e^{-i\nu\eta}, \tag{4.31}$$

which is not a reflected wave but just noise as in (4.7). Its coefficient need not be calculated, as it emerges that it does not affect even $\mathscr{R}_{42}(\tau)$.

In part 1 we showed that $\mathscr{T}_{31}(\tau) = O(e^{-\nu\pi})$. As this is an asymptotic analysis for large ν the exponentially small terms cannot be calculated but it is fairly clear from the method used here that $\mathscr{T}_{31}$ will be smaller than any negative power of η, since it is the exponents that matter and they have all been included.

So far the position is as follows. At large values of t ($\gg \nu$) we have a quasi-steady situation with a primary incident wave that is absorbed at $\alpha U'(0)\, yt/\nu = 1$ ($\eta = 1$), its only effect on the region $\eta < 1$ being to produce critical-layer noise that decays as $\eta \to -\infty$. As τ increases this wave starts to be reflected from the line $\eta = \eta_0$ ($= \frac{1}{4}(1+\sqrt{5})^2$) with a reflection coefficient that is $O(\nu^{-1}\tau^{3+2i\nu})$. Only critical-layer noise is generated in $\eta < \eta_0$. In § 5 we show that a second harmonic is generated above the line $\eta = 2+\sqrt{3}$ ($> \eta_0$) which gives $\mathscr{R}_{42}(\tau)$. An additional noise term is generated below $\eta = \frac{1}{3}(4+\sqrt{7})$ ($< \eta_0$).

The results of this section have also been checked using an alternative method in which we write $G_{31} = \tau^{\frac{5}{2}+i\nu}L_{31}(\eta)$ in (2.16), (2.17), and construct the solution regular at $\eta = 0$ with the correct behaviour as $|\eta| \to \infty$ by the method of variation of parameters. Examination of the behaviour of the integrals for $\nu \gg 1$, which we shall not reproduce, shows that, at any value of η at which there is no saddle point, L_{31} is given by (4.9) multiplied by the appropriate exponential, but at a saddle point it is larger by a factor $\nu^{-\frac{1}{2}}$ and is given by a constant multiple of a complementary function as in (4.18). The saddle points of the integrals correspond to the zeroes of $Q_{31\lambda}(\eta)$.

Consideration of the equations for Ψ_{33} shows that no complementary functions are generated and the terms are obtained by division of the forcing terms by the appropriate $\nu Q_{33\lambda}(\eta)/\eta$ as in (4.9). Thus there is no induced third harmonic that is $O(\tau^{\frac{5}{2}+3i\nu})$ and no reflection or transmission at this stage.

5. The calculation of $\mathscr{R}_{42}(\tau)$

Continuing with the expansion we now consider the calculation of Ψ_4 and S_4 in (2.9). Among the various constituents of these functions in (2.10), it is easy to show that Ψ_{44}, S_{44} arise entirely from terms similar to (4.3) with $r = n = 4, 0 \leqslant \lambda \leqslant 4$, and merge into the general noise structure in the shear layer; no complementary functions are induced. Also Ψ_{40}, S_{40} modify the basic shear flow in a similar way as do Ψ_{20}, S_{20}. Interest at this stage therefore centres chiefly on the computation of Ψ_{42}, S_{42} and we shall show here that there is a reflected wave among their components which provides the leading term in the second harmonic of the reflection coefficient $\mathscr{R}$.

There are three groups of terms in Ψ_{42}; one group arises from the interaction of the induced complementary functions (4.18) of G_{31}^{+} with Ψ_{11}, S_{11}, another is formed by direct forcing from Ψ_{11} and is given by terms of the type (4.3), and the third consists of induced complementary functions of $G_{42}^{\pm}$. Let us dispose of the first of these groups. They are present only when $\eta > \eta_0$ $(=\frac{1}{4}(1+\sqrt{5})^2)$ and are forced by terms with exponential behaviour:

$$e^{i\nu\eta}\,\eta^{-3i\nu}\,e^{i\nu D_{31}(\eta)}, \quad \eta^{-2i\nu}\,e^{i\nu D_{31}(\eta)}, \quad e^{-2i\nu\eta}\,e^{-i\nu D_{31}(\eta)}, \quad e^{-3i\nu\eta}\,\eta^{i\nu}\,e^{-i\nu D_{31}(\eta)}. \tag{5.1}$$

The corresponding contributions to $G_{42}^{\pm}$ are obtained from these on multiplying by algebraic functions of η that are bounded for all $\eta > \eta_0$ and tend to zero as $\eta \to \infty$. Hence they generate no new complementary functions nor any reflected wave.

The forcing terms coming directly from Ψ_{11}, S_{11} are all of the form (4.3) with $r = 4$, $n = 2$, and $-1 \leqslant \lambda \leqslant 3$ for $\eta > 1$ while $\lambda = 2$ if $\eta < 1$. We now make the substitution (4.4) and consider (4.5) with $r = 4, n = 2$. The complementary functions of these equations take the asymptotic forms given in (4.7), (4.8). Further it is impossible to select any complementary function which does not represent an incoming wave from either above or below the shear layer or which is singular at $\eta = 0$. Hence no complementary functions can be generated over the entire range of η. They can only appear for a range of η bounded above or below.

For $\nu \gg 1$ a particular integral of (4.5) is given by

$$T_{42}^{\pm} \sim \frac{\eta\Phi_{42\lambda}^{\pm}(\eta)}{\nu[\lambda(2-\lambda)\,\eta^2+2(\lambda^2-2\lambda+2\pm 1)\,\eta+\lambda(2-\lambda)]}, \tag{5.2}$$

and we deduce that this gives the required contribution to Ψ_{42} unless the denominator $Q_{42\lambda}(\eta)$ of (5.2) vanishes in $\eta > 1$; for $\eta < 1$ we have $\lambda = 2$, and $Q_{42\lambda}$ vanishes only at $\eta = 0$ which is not a singularity of T_{42}. The denominator has a zero in $\eta > 1$ in two cases only. These are $\lambda = -1$ and $\lambda = 3$, and in each case $Q_{42\lambda}^{+}$ vanishes at

$$\eta_2^{+} = 2+\sqrt{3}, \tag{5.3}$$

and $Q^-_{42\lambda}$ at

$$\eta^-_2 = \tfrac{1}{3}(4+\sqrt{7}). \tag{5.4}$$

We shall now show that when $\lambda = -1$ the singularity in T^+_{42} at $\eta = \eta^+_2$ induces a complementary function for $\eta > \eta^+_2$ that gives rise to a reflected wave with reflection coefficient $\mathscr{R}_{42}$, while that at η^-_2 in T^-_{42} induces a complementary function in $\eta < \eta^-_2$ corresponding to CL-noise. Nevertheless, this is a departure from previous results in that the solution in $\eta < 1$ is no longer composed solely of terms like those in (4.3), and eventually the modifications induced lead to a transmitted wave. When $\lambda = 3$ the complementary functions appear only for $\eta > \eta^\pm_2$ and correspond to noise when $\eta \gg 1$. Again there is no transmission coefficient at this stage though, as pointed out in the discussion, one is expected to be associated with Ψ_{91}.

We first consider $\lambda = -1$ and obtain $\mathscr{R}_{42}(\tau)$. This arises from T^+_{42} and is computed in much the same way as $\mathscr{R}_{31}(\tau)$ in § 4. The equation corresponding to (4.13) is

$$\frac{\eta}{\nu}\frac{d^2T^+_{42}}{d\eta^2} + \left\{4i(\eta-1)+\frac{5}{\nu}\right\}\frac{dT^+_{42}}{d\eta} + \left\{-\frac{3\nu}{\eta}(\eta^2-4\eta+1)+4i-\frac{12i}{\eta}\right\}T^+_{42} = \Phi_{42,-1}(\eta). \tag{5.5}$$

We now compute $\Phi_{42,-1}$. Since the term in $\eta^{-2i\nu}$ in Ψ_{22} turned out to be ν^{-1} smaller than expected, its effect as a forcing term can be ignored. This means that Ψ_{33} can also be neglected as we are computing only the leading term for $\nu \gg 1$. We need the terms $O(\eta^{-2i\nu}e^{i\nu\eta})$ in Ψ_{31}, S_{31}, which are, respectively, to leading order in ν,

$$-\tfrac{1}{2}A|A|^2\tilde{\beta}\,\tau^{\frac{5}{2}+i\nu}e^{i\nu\eta}\,\eta^{-2i\nu}\frac{\eta^3-\eta^2-3\eta+1}{\nu^{\frac{5}{2}}(\eta-1)^3(\eta^2-3\eta+1)}, \tag{5.6}$$

$$-\tfrac{1}{2}A|A|^2\tilde{\beta}\,\tau^{\frac{7}{2}+i\nu}e^{i\nu\eta}\,\eta^{-2i\nu}\frac{(\eta-2)(\eta^3-5\eta^2+9\eta-3)}{\nu^{\frac{7}{2}}\eta(\eta-1)^3(\eta^2-3\eta+1)}. \tag{5.7}$$

Thus the forcing term $-\nu(M_{42}+i\nu H_{42})$ for the equation for G^+_{42} has the contribution

$$-iA^2|A|^2\tilde{\beta}\,\tau^{4+2i\nu}e^{i\nu\eta}\,\eta^{-3i\nu}\frac{(\eta-1)(\eta-2)}{(\nu\eta)^{\frac{3}{2}}(\eta^2-3\eta+1)}. \tag{5.8}$$

Then finally, to leading order in ν,

$$\Phi_{42,-1}(\eta) \sim -iA^2|A|^2\tilde{\beta}\frac{(\eta-1)(\eta-2)}{(\nu\eta)^{\frac{3}{2}}(\eta^2-3\eta+1)}, \tag{5.9}$$

with an error that is of relative order $\nu^{-\frac{1}{2}}$.

As in the calculation of $\mathscr{R}_{31}$ the complementary function must be added for $\eta > \eta^+_2$ $(=2+\sqrt{3})$ and is written as $C_{42}(\eta)\,e^{i\nu D_{42}(\eta)}$, where

$$D_{42}(\eta) = -2\eta+3\log\eta+(\eta^2+4\eta+1)^{\frac{1}{2}}+2\log\{\eta+2+(\eta^2+4\eta+1)^{\frac{1}{2}}\} \\ -\log\{1+2\eta+(\eta^2+4\eta+1)^{\frac{1}{2}}\}, \tag{5.10}$$

$$C_{42}(\eta) = C_1\frac{\{\eta+2+(\eta^2+4\eta+1)^{\frac{1}{2}}\}^3}{(\eta^2+4\eta+1)^{\frac{1}{4}}\{1+2\eta+(\eta^2+4\eta+1)^{\frac{1}{2}}\}^2}, \tag{5.11}$$

and C_1 is a constant. It is noted that D_{42} is similar in form to D_{31}, having a removable singularity at $\eta = 0$, so that G_{42} is regular there, and two transition points at

$$\eta = -2 \pm \sqrt{3}.$$

As we argued in § 4, unless $C_1 = 0$ for $\eta < \eta^+_2$ an unacceptable, exponentially large, incident wave appears as $\eta \to -\infty$. Finally, the behaviour of Ψ_{42} for $\eta \gg 1$ is found to be

$$\Psi_{42} \sim -iA^2|A|^2 e^{3i\nu} \frac{2^{5i\nu+2}\,\tau^{4+2i\nu}\,\eta^{\frac{1}{2}+i\nu}}{3^{\frac{21}{4}+3i\nu}(1+\sqrt{3})^{6i\nu+1}\,\nu^3}, \tag{5.12}$$

which leads to $\mathscr{R}_{42}(\tau)$. The result is, with a relative error $O(\nu^{-\frac{1}{2}})$,

$$\mathscr{R}_{42}(\tau) \sim -\frac{2^{5i\nu+2}\,\epsilon^{-2i\nu}\,e^{3i\nu(1-\Gamma^+)}\,i}{3^{\frac{21}{4}+3i\nu}(1+\sqrt{3})^{6i\nu+1}\,\nu^{\frac{3}{2}+3i\nu}}\left(\frac{R'(\infty)}{R'(0)}\right)^{\frac{3}{4}}\frac{(U'(0))^{3+3i\nu}}{(U(\infty))^{\frac{3}{2}}}\,\tau^{\frac{9}{2}+3i\nu}. \tag{5.13}$$

This is the first coefficient of $e^{i(2\alpha x+m_+y)}$ induced outside the critical layer as τ increases. Here, as in (2; 4.7),

$$m_+ = \nu(R'(\infty)/R'(0))^{\frac{1}{2}}\,(U'(0)/U(\infty)). \tag{5.14}$$

We now examine briefly the effect on T_{42}^+ of the singularity at $\eta = \eta_2^+$ with $\lambda = 3$, and on T_{42}^- of the singularities at $\eta = \eta_2^-$ when $\lambda = -1$ and 3. For the effect on T_{42}^+ with $\lambda = 3$ we find that the complementary function which must be added for $\eta > \eta_2^+$ has exponential behaviour $e^{-i\nu D_{42}(\eta)}$, where D_{42} is given by (5.10). Thus the product

$$\eta^{i\nu}\,e^{-3i\nu\eta}\,e^{-i\nu D_{42}(\eta)} \sim \eta^{-3i\nu}\,e^{-2i\nu\eta} \tag{5.15}$$

for $\eta \gg 1$, and we have the decaying second complementary function as in (4.7). It is easy to show that for large η there is a factor $\eta^{-\frac{11}{2}}$ in (5.15).

For T_{42}^- we first consider $\lambda = -1$. There is a boundary layer at $\eta = \eta_2^-$ $(= \frac{1}{3}(4+\sqrt{7}))$ that induces a complementary function with exponential behaviour $e^{i\nu F_{42}(\eta)}$, where

$$F_{42}(\eta) = -2\eta + 3\log\eta + (\eta^2+1)^{\frac{1}{2}} - \log\{1+(\eta^2+1)^{\frac{1}{2}}\}. \tag{5.16}$$

The product $\eta^{-3i\nu}\,e^{i\nu\eta}\,e^{i\nu F_{42}(\eta)}$ is regular at $\eta = 0$, and for $\eta \gg 1$ is of the form $\eta^{-i\nu}$ so would contribute a term in $\tau^{4+2i\nu}\eta^{\frac{1}{2}-i\nu}$ to G_{42}^-, which is unacceptable. However, for $\eta \ll -1$ the product is of the form $(-\eta)^{-i\nu}\,e^{-2i\nu\eta}$, and corresponds to the second complementary function in (4.8) which represents decaying CL-noise. This is acceptable and therefore a multiple of the complementary function must be added for $\eta < \eta_2^-$. When $\lambda = 3$ a similar argument shows that the complementary function must be included for $\eta > \eta_2^-$, but for $\eta \gg 1$ it decays as noise.

6. The strategy of the expansion

In this section we shall explain in general terms the manner in which the leading terms of the principal harmonic and induced second harmonic of the reflected wave appeared, and indicate the subsequent development of the solution in ascending real powers of τ. The crucial equations are (2.16), (2.17) for $G_{rn}^\pm$, and since, as noted in (4.3), the τ-dependence of $\Psi_{rn}(\tau,\eta)$ is $\tau^{-2+ni\nu+\frac{3}{2}r}$, we write

$$G_{rn}^\pm(Y,\tau) = \tau^{-2+ni\nu+\frac{3}{2}r}\,P_{rn}^\pm(\eta), \tag{6.1}$$

whence $P_{rn}^\pm$ satisfy

$$\frac{\eta}{\nu}\frac{d^2P_{rn}^\pm}{d\eta^2} + \left\{ni(\eta+1) + \frac{3r-2}{2\nu}\right\}\frac{dP_{rn}^\pm}{d\eta} - ni(\tfrac{1}{2} \pm i\nu)\,P_{rn}^\pm = \Sigma f_{rn}^\pm\,e^{i\nu E_{rn}^\pm}. \tag{6.2}$$

Here $E_{rn}^\pm$ are functions of η to be specified more precisely below and $f_{rn}^\pm$ are also functions of η and ν which may be written in descending powers of ν. We have obtained the values of E and f in special cases earlier in this paper. The summation sign in (6.2) indicates that at any stage of the expansion there are a finite number of such terms, of similar form but with different Es.

Any reflected wave derives from one of the complementary functions of the equation for P^+_{rn} when $\eta \gg 1$ and any transmitted from the equation for P^-_{rn} for $\eta \ll -1$. Similarly, when $\eta \ll -1$ the equation for P^+_{rn} contains unacceptable waves incident from below the shear layer, and when $\eta \gg 1$ that for P^-_{rn} contains waves incident from above the layer, of which Ψ_{11} is the only one to be permitted. When $\nu \gg 1$ the exponential behaviour of these complementary functions is

$$e^{i\nu U^{\pm}_n(\eta)}, \quad e^{i\nu V^{\pm}_n(\eta)}, \tag{6.3}$$

where, from (6.2),

$$U^{\pm\prime}_n(\eta) = \frac{1}{2\eta}\{-n(\eta+1)+[n^2(\eta+1)^2 \pm 4n\eta]^{\frac{1}{2}}\}, \tag{6.4}$$

$$V^{\pm\prime}_n(\eta) = \frac{1}{2\eta}\{-n(\eta+1)-[n^2(\eta+1)^2 \pm 4n\eta]^{\frac{1}{2}}\}. \tag{6.5}$$

We examine now the principal properties of the complementary functions for P^+_{rn} defined by U^+_n and V^+_n. When $\eta \gg 1$, U^+_n gives the reflected wave since $U^{+\prime}_n \approx \eta^{-1}$, and $V^{+\prime}_n \approx -n$, which corresponds to noise. Moreover, $V^{+\prime}_n$ is singular at $\eta = 0$ and both $U^{+\prime}_n$ and $V^{+\prime}_n$ have singularities at the (negative) roots $\eta_{n1}, \eta_{n2}\,(\eta_{n1} > \eta_{n2})$ of

$$n^2(\eta+1)^2 + 4n\eta = 0.$$

If either of the associated complementary functions is non-zero in $0 > \eta > \eta_{n1}$ it splits into two branches at $\eta = \eta_{n1}$, one of which becomes larger by a factor $e^{\nu\pi}$ at $\eta = \eta_{n2}$, the other being smaller by the same factor. At $\eta = \eta_{n2}$ a further split occurs, and for $\eta < \eta_{n2}$ both exponential behaviours in (6.3) occur, of which V^+_n represents a wave incident from below the shear layer.

The complementary functions for P^-_{rn} are simpler since, for $n > 1$, $n^2(\eta+1)^2 - 4n\eta$ has no real roots. Thus U^-_n gives the exponential behaviour of the complementary function regular at $\eta = 0$ for all η, and represents an incident wave for $\eta \gg 1$ and noise for $\eta \ll -1$. On the other hand V^-_n represents a transmitted wave for $\eta \ll -1$ and noise for $\eta \gg 1$ but is singular at $\eta = 0$. If $n = 1$

$$U^{-\prime}_1(\eta) = \frac{1}{2\eta}\{-(\eta+1)+|\eta-1|\}, \quad V^{-\prime}_1(\eta) = \frac{1}{2\eta}\{-(\eta+1)-|\eta-1|\}, \tag{6.6}$$

and it should be understood that either complementary function splits into a linear combination of both on crossing the point $\eta = 1$.

These complementary functions appear in the expansion of Ψ, S in the following way. A particular integral of (6.2) for $\nu \gg 1$ (not necessarily uniformly valid for all η) can be obtained by assuming that $P^{\pm}_{rn}$ has the same exponential behaviour as the right-hand side and, provided this particular integral is not singular, no complementary function is needed. A singularity arises if

$$W'_n(\eta) = E'_{rn}(\eta), \tag{6.7}$$

for some η, say η_s where W_n is any one of $U^{\pm}_n$, $V^{\pm}_n$. Then, depending on the value of η_s, one of the two possible complementary functions appears, for $\eta > \eta_s$ or for $\eta < \eta_s$. The criterion for selection is simply that a complementary function singular at $\eta = 0$

is not permitted in any range including $\eta = 0$; and complementary functions representing incident waves are not allowed in $\eta \gg 1$ or $\eta \ll -1$. Thus if we consider the contribution to the right-hand side of (6.2) which comes directly from Ψ_{11}, S_{11}, i.e. if we neglect the effect of any other complementary functions with $n > 1$,

$$E_{rn}^{\pm} = -\lambda\eta - (n-\lambda)\log\eta = (n-\lambda)\,U_1^-(\eta) + \lambda V_1^-(\eta), \tag{6.8}$$

where $-\frac{1}{2}(r-n) \leqslant \lambda \leqslant \frac{1}{2}(r+n)$. For $\eta < 1$

$$E_{rn}^{\pm}(\eta) = -n\eta = nU_1^-(\eta). \tag{6.9}$$

This last equation implies that (6.7) has no solution for any $\eta < 1$ except if $n = 1$ and $W = U_1^-$. Then it becomes an identity, but as a result no new complementary function is generated.

For $\eta > 1$, complementary functions arise if, for any η,

$$-\lambda - \frac{n-\lambda}{\eta} = \frac{1}{2\eta}\{-n(\eta+1) \pm [n^2(\eta+1)^2 \pm 4n\eta]^{\frac{1}{2}}\}. \tag{6.10}$$

Thus we see that if $\lambda < 0$, $U_n^{\pm}$ can occur, while if $n < \lambda$, $V_n^{\pm}$ can occur, except when $n = 1$. When $n = 1$, $E_{r1}^{-\prime}(1) = V_1^{-\prime}(1) = U_1^{-\prime}(1)$. We conclude that if $n = r, r > 1$ no complementary functions are induced by (6.6) while if $r > n > 1$ then U_n^+ is induced for $\eta > \eta_{s1}\,(>1)$ (the appropriate solution of (6.10)), U_n^- is induced for $\eta < \eta_{s2}$ and $V_n^{\pm}$ are induced for $\eta > \eta_{s3}, \eta_{s4}$, the values of η being different in all four cases. The situation is similar if $n = 1$ for U_1^+, V_1^+, but for U_1^-, V_1^- special care must be taken since $\eta_s = 1$ in both instances. The treatment of the solution near $\eta_s = 1$ has already been considered in § 4, and again we may conclude that no additional complementary function is in fact induced by the double singularity which results.

As explained earlier, each U_n^+ corresponds to a reflected wave for $\eta \gg 1$, each $V_n^{\pm}$ corresponds to noise for $\eta \gg 1$ and each U_n^- corresponds to noise for $\eta \ll -1$. Thus we can see in principle that reflected waves of all harmonics occur, the leading term of the nth harmonic appearing at the $(n+2)$th stage and being of amplitude $\tau^{\frac{3}{2}(n+1)}$. Furthermore, after the $r = 3$ stage not all the E_{rn} in (6.2) come from (6.8) if $\eta > 1$. We may plausibly expect that E_{rn} will in general be a linear combination of U_1^-, U_s^+, $V_s^{\pm}$ $(1 \leqslant s \leqslant r-2)$ with the provisos that the sum of their coefficients, after multiplication by s, be equal to n, and to r if no regard is paid to sign. We shall discuss the general situation in $\eta < 1$ in § 7.

7. The transmitted wave

We have already seen that for $r \leqslant 3$ the only possible form of E_{rn} when $\eta < 1$ is $-n\eta$, but for $r \geqslant 4$ additional terms must be considered since the restraint on the incident wave implies that $U_n^-(\eta)$ can appear in some of the E_{rn} when $\eta < \eta_{s2}$. In order to induce a transmitted wave, a complementary function with V_n^- as exponential factor must be generated in $\eta < 0$; were one to be generated in $\eta > 0$ the singularity at $\eta = 0$ in V_n^- ensures that it would be confined to an infinite region above $\eta = 0$. Now we have already seen that the simplest form of E_{rn} for $\eta < 0$, namely nU_1^-, cannot generate a V_n^- and so we must now consider E_{rn} which involve U_n^- for $n > 1$

and which can occur if $r \geqslant 4$. Two inequalities are of assistance in this study, namely that, when $\eta < 0$,

$$\left.\begin{aligned} U_n^{-\prime}(\eta) < U_{n-1}^{-\prime}(\eta) < 0, \quad nU_{n-1}^{-\prime}(\eta) < (n-1)\,U_n^{-\prime}(\eta) < 0, \\ V_n^{-\prime}(\eta) > V_{n-1}^{-\prime}(\eta) > 0, \quad nV_{n-1}^{-\prime}(\eta) > (n-1)\,V_n^{-\prime}(\eta) > 0. \end{aligned}\right\} \tag{7.1}$$

As a preliminary, we observe that, as shown in § 6, U_n^- is not induced by the computation for $n = r$, and hence E_{rn} does not contain U_{r-2}^- if $n = r-2$. Otherwise, we may see by inspection that a typical form for E'_{rn} is given by

$$\sum_1^{S_0} \lambda_s U_s^{-\prime} - \sum_1^{T_0} \mu_t U_t^{-\prime}, \quad \lambda_{S_0} \neq 0, \tag{7.2}$$

where $\lambda_s \geqslant 0, \mu_t \geqslant 0$. There are restrictions on S_0, T_0, namely that $T_0, S_0 \leqslant r-3$, and more importantly

$$\sum_1^{S_0} s\lambda_s - \sum_1^{T_0} t\mu_t = n, \tag{7.3}$$

$$\sum_1^{S_0} s\lambda_s + \sum_1^{T_0} t\mu_t = Z_{rn}, \tag{7.4}$$

where $Z_{rn} = r$ if $S_0 = T_0 = 1$ but $Z_{rn} \leqslant r-2$ otherwise. Then by the inequalities of (7.1) it follows that

$$\begin{aligned} E'_{rn} \leqslant \frac{U_{S_0}^{-\prime}}{S_0} \sum_1^{S_0} s\lambda_s - U_1^{-\prime} \sum_1^{T_0} t\mu_t &= nU_1^{-\prime} + \tfrac{1}{2}\left(\frac{U_{S_0}^{-\prime}}{S_0} - U_1^{-\prime}\right)(Z_{rn}+n) \\ &\leqslant nU_1^{-\prime} + \tfrac{1}{2}(r-2+n)\left(\frac{U_{S_0}^{-\prime}}{S_0} - U_1^{-\prime}\right). \end{aligned} \tag{7.5}$$

Since $S_0^{-1} U_{S_0}^{-\prime} - U_1^{-\prime}$ is positive and an increasing function of S_0 we maximize the last expression if we choose S_0 to be as large as possible for fixed r, n. This by inspection is $\frac{1}{2}(r+n)-1$, so finally

$$E'_{rn} \leqslant U_{S_0}^{-\prime} + (n - S_0)\, U_1^{-\prime}. \tag{7.6}$$

We now see that the best chance of generating a complementary function involving V_n^- at the rth stage of the expansion is if

$$E'_{rn} = U_{S_0}^{-\prime} + (n-S_0)\, U_1^{-\prime}, \quad S_0 = \tfrac{1}{2}(r+n)-1, \tag{7.7}$$

except when $n = r-2$ or r, in which case we know that it is impossible; E_{rr} generates no complementary functions and $E_{r,r-2}$ generates only U_{r-2}^-. In order to generate V_n^- in $\eta < 0$ we must have

$$V_n^{-\prime}(\eta) = U_{S_0}^{-\prime}(\eta) + (n-S_0)\, U_1^{-\prime}(\eta), \tag{7.8}$$

for some $\eta < 0$. Since V_n^- has a singularity at $\eta = 0$, the complementary function is non-zero only for more negative values of η. By direct substitution it may be shown that (7.8) reduces to

$$nS_0\eta^2 + \{n^2 + S_0^2 - 2(n+S_0) + 1\}\eta + nS_0 = 0, \tag{7.9}$$

which has real negative zeroes only if

$$S_0 \geqslant (1 + n^{\frac{1}{2}})^2. \tag{7.10}$$

It now follows that the first appearance of a V_n^- complementary function in $\eta < 0$, and hence of a transmitted wave, is associated with $n = 1$ and occurs at the *ninth* stage of the expansion (2.9). The appropriate value of S_0 is 4 and (7.9) has a double zero at $\eta = 1$. In view of the double zero in the quadratic corresponding to (4.8), the transition layer near $\eta = -1$ is of thickness $\nu^{-\frac{1}{3}}$ instead of $\nu^{-\frac{1}{2}}$, which would be the case in the neighbourhood of a simple zero, and this permits a discontinuity in the V_1^- complementary function across it. A V_n^- complementary function cannot occur for $\eta > -1$, since it would be singular at $\eta = 0$, and therefore must be set up when $\eta < -1$, leading to a transmitted wave as $\eta \to -\infty$ except in the fortuitous and highly unlikely case of the forcing function vanishing to a large enough order at $\eta = -1$. We conclude that the transmission coefficient has amplitude $O(\tau^{12})$ in the first harmonic.

For the second harmonic the roots of (7.9) are real for $S_0 \geqslant 6$ and when $S_0 = 6$ they occur at $\eta = -\frac{3}{4}, -\frac{4}{3}$. Thus for $S_0 = 6$ two complementary functions are set up at the twelfth stage, and except if they happen to cancel each other exactly when $\eta < -\frac{4}{3}$ we obtain a second harmonic to the transmission coefficient of order $\tau^{\frac{33}{2}+i\nu}$. Higher harmonics do not occur before the $(4+4n^{\frac{1}{2}}+n)$th stage.

8. Discussion

According to linear theory a strongly stratified shear layer has the property, first discovered by Booker & Bretherton (1967), of absorbing an incident wave of amplitude $\epsilon\,(\ll 1)$ so that it is neither reflected nor transmitted. Our aim in the three papers of this work was to elucidate further the nature of the absorption process and to examine whether the same properties persist indefinitely. We were not concerned with flow properties $O(\epsilon^2)$ which might arise from the effect of nonlinear terms in the governing equations at a general point in the flow field and may well depend on the precise way in which the incident wave and the shear flow are set up. We concentrated our study on phenomena of the same size as the incident wave but which, being generated by nonlinear terms, might take a long time to evolve.

In the first part we showed that such effects arise from the neighbourhood of the critical level at which the phase velocity of the incident wave and the local fluid velocity are equal, conveniently taken to be $y = 0$. A general procedure for their calculation has been developed which, in principle, would enable us to compute the shapes of the reflected and the transmitted waves at this level and for all local Richardson numbers $J\,(= \nu^2 + \frac{1}{4})$. However, it rapidly becomes very cumbersome, and we contented ourselves with the computation of the leading term in the reflected wave when $\nu \gg 1$ and to noting that the corresponding term in the transmitted wave is exponentially small. In a separate study Mr A. T. Burke has removed the restriction $\nu \gg 1$ from this calculation. The complexity of the work involved also obscures the significance of the linear mechanics of the absorption process and of the generation of the reflected wave.

Consequently in part 2 we made a fresh start by assuming that $\nu \gg 1$ from the outset and obtaining explicit expressions for the linear solution at large times. We showed that the linear equations in the shear layer contain two pairs of solutions, one of which matches to an external wave motion and the other has an algebraic dependence on time $(t^{-\frac{1}{2}+i\nu})$, and which may conveniently be described as noise. This second class is closely related to the genesis of the disturbance and so any phenomenon which depends on its properties must be regarded as being to some extent arbitrary and not easily susceptible to study by the methods of this paper. Fortunately such effects are $O(\epsilon^2)$, except in the neighbourhood of this critical level where the noise is virtually independent of initial conditions and also increases in intensity to become as important as the wavelike solutions. Indeed, were the wave allowed to penetrate to the critical level at $y = 0$, an unacceptable singularity would occur and this is prevented by a transition near $\alpha U'(0)\, yt = \nu$, where it is converted into critical-level noise.

These two classes of solution take on particularly simple forms near $y = 0$, when $\nu \gg 1$, and these are also representative of their complete asymptotic expansions in descending powers of ν. Hence their use in the nonlinear theory enabled us to obtain explicit expressions for the dominant part of the various terms of the reflection and transmission coefficients when $\nu \gg 1$, more simply than by the precise method of part 1, and moreover we could comment in more detail on their general properties. In part 3 we have developed the nonlinear solution as a series in powers of $\epsilon^{\frac{2}{3}}\alpha t\,(=\tau)$ through the use of the nonlinear terms of the equations as sources for the linear equations. In general, these sources generate solutions of the same form as the sources and are then of no significance, being $O(\epsilon^2)$ outside the critical layer. However, in certain circumstances such solutions become singular at one or two values of $\eta\,(=\alpha U'(0)\, yt/\nu)$, typically $\eta = \eta_s$, and in order to have a smooth solution a complementary function must be generated either in $\eta > \eta_s$ or $\eta < \eta_s$, depending on whether this function is singular at $\eta = 0$ or corresponds to an unacceptable behaviour outside the shear layer (e.g. is an incident wave from below).

Moreover, the appearance of a wave outside the shear layer depends on the class of complementary function so forced. A complementary function of the other class merely gives rise to noise at this stage, although at later stages it will contribute to the nonlinear sources and lead to a singularity which provokes the wavelike solution.

It can be seen, almost at once, that such must be the case for the transmitted wave, since the linear solution of part 2 is very simple in $\eta < 1$ and the associated nonlinear sources cannot give rise to any additional complementary functions in $\eta < 0$. The situation with the reflected wave is quite different, a whole succession of new complementary functions being generated above the critical level as the expansion proceeds and, either directly or indirectly, they give rise to reflected waves. Outside the shear layer the form ψ_{R} taken by the reflected wave is

$$\psi_{\mathrm{R}} = \sum_{n=1}^{N} \sum_{l=0}^{\infty} c_{nl}(\nu) \left\{ \frac{U'(0)}{(U(\infty))^{\frac{1}{2}}} \left(\frac{R'(\infty)}{R'(0)} \right)^{\frac{1}{4}} \epsilon(\alpha t)^{\frac{3}{2}} \right\}^{n+2l+1} \cos\{n\alpha x + m_n y + d_{nl}(\nu, t)\}, \tag{8.1}$$

where d_{nl} are algebraic functions of ν, $\log \nu$ and $\log t$, and c_{nl} are algebraic functions of ν, of which

$$c_{10} \approx 0{\cdot}065\nu^{-1}, \quad c_{20} \sim 0{\cdot}009\nu^{-\frac{3}{2}}. \tag{8.2}$$

Also

$$m_n = +\left\{ \frac{U'^2(0)}{U^2(\infty)} \frac{R'(\infty)}{R'(0)} (\nu^2 + \tfrac{1}{4}) - n^2\alpha^2 \right\}^{\frac{1}{2}}, \tag{8.3}$$

and N is the maximum value of n for which $m_n^2 > 0$. Thus further harmonics occur in the reflected wave. A parallel situation is noted for unstratified shear flows with a mean vorticity gradient at the critical layer (Stewartson 1978). In that case there is a velocity jump across the layer that takes the form

$$-\pi \sin x + \pi \sum_{n=1}^{\infty} \sum_{l=0}^{\infty} b_{nl} (\epsilon^{\frac{1}{2}} t)^{4l+2n+2} \sin nx, \tag{8.4}$$

with

$$b_{10} = \tfrac{1}{192}, \quad b_{20} = -\tfrac{1}{5760}, \tag{8.5}$$

which indicates orders of magnitude comparable with (8.2) for the leading numerical terms. In the simplest problem of this type Stewartson was able to show that the velocity jump tended to zero as t tended to infinity.

In order to generate a transmitted wave it is first necessary to modify the structure of the solution in $\eta < 1$ from that given in part 2. This necessitates a new complementary function forced by a singularity of the nonlinear-source solution at a value $\bar{\eta}_s (> 1)$ of η which can only be non-zero if $\eta < \bar{\eta}_s$. (The reflected wave comes from complementary functions non-zero in an infinite interval of η bounded from below.) This first happens at the fourth stage of the expansion, and the complementary function is a second harmonic of the original wave but it is not sufficiently different from the principal harmonic to produce the required singularity in $\eta < 0$. We have to wait until the *sixth* stage, when a noise-like complementary function, which is a fourth harmonic, is generated in $\eta < \bar{\eta}_s$. Then at the *ninth* stage the interaction of this solution with three principal harmonic solutions produces a nonlinear source, which gives rise to a singularity in $\eta < 0$, and this provokes a principal harmonic of the transmitted wave. After that, higher harmonics appear in an orderly fashion and also additional contributions to the lower harmonics are made; we have for the transmitted wave ψ_T outside the shear layer the form

$$\psi_T = f_{11}(\epsilon^{\frac{2}{3}}\alpha t)^{12} \cos(\alpha x + m_1 y + g_{11}) + \dots, \tag{8.6}$$

where f_{11} is a function of ν only and has not yet been calculated.

The solution in this paper is determined on the assumptions that $\epsilon \ll 1, \nu \gg 1$, and that $\epsilon^{\frac{2}{3}}t$ is finite but small. A study of (8.1) suggests that this last assumption is unnecessarily strong and the correct requirement is that $\epsilon^{\frac{2}{3}}t \ll \nu^b$ where b is a positive number. It is not easy to decide on the precise value of b in view of the resonances, which increase the order of magnitude of the induced complementary function by $\nu^{\frac{1}{2}}$, or even more in the case of double resonance, but from (8.1) we may anticipate that $b \simeq \frac{1}{3}$. Now the asymptotic forms (2; 5.16), (2; 5.17) of the linearized solution are crucial to the success of the method and these are valid if $t \gg \nu$. Hence we must have

$$1 \ll (t/\nu)^{\frac{2}{3}} \ll (\epsilon\nu)^{-1} \tag{8.7}$$

on taking $b = \frac{1}{3}$.

Here only the leading terms in the expansions for $\nu \gg 1$ have been discussed, and even so f_{11} in (8.6) was not found explicitly. Since the further terms are of an essentially similar kind we can be sure that the error in (8.1) is exponentially small and probably $O(e^{-\nu\pi})$, which is the order of magnitude consistently neglected throughout the paper. It is of interest to consider the applicability of the present results to finite values of the Richardson number since $e^{-\nu\pi}$ is very small even when $\nu = 1$. On the time scale on which the results of this study are valid the distortion to the mean flow is only $O(\epsilon^2\nu^2)$ and hence small, so it seems unlikely that the local Richardson number will fall so low as to cause instability before the reflection and transmission predicted here have taken place. Indeed it is not clear that a Richardson number of less than 0·25 is a sufficient condition for dynamic instability. Geller, Tanaka & Fritts (1975) use it as a criterion, but Brown & Stewartson (1978) showed that the instability of a marginally unstable stratified shear flow was determined by the second and higher harmonics. The latter theory cannot be extended to stable basic flows because there are no eigensolutions of the right kind. This difficulty also prevents us from making precise statements for the present problem.

Laboratory experiments on critical layers are difficult to perform but a recent one by Thorpe (1981) is of especial interest to us. The mean flow was produced by tilting a tube containing stratified fluid and forcing transient waves by corrugating its base. The effective Richardson number was initially about unity and fell to unstable values after a few seconds. Nevertheless, Thorpe was able to distinguish some signs of wave reflection but not of transmission.

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